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The Role of Representation(s) in Developing Mathematical Understanding

ELLEN, A 4-YEAR-OLD CHILD, IS PLAYING with several toys. As she sets the toys out to play, she names the first as *one*, the second as *two*, and so on. These words are symbols for the position of the toys in the series the child is enumerating. In the beginning, she may not understand that the last number stated is the number of toys all together. These number words may simply be words the child has learned to utter as she touches each object in a series of objects. Later, at around 5 or 6 years old, the child begins to understand that the last number named in this game is the number of toys that are in the set and finally that there is a numeral that represents the number of elements in the set. Once learned, the numeral and its name (e.g., *5* and *five*) become the external representations that are the conventions for the internal abstraction, the number of elements in the set. Thus, the number name, *five*, and the numeral, *5*, are the external representations that act to stimulate an image, the internal representation, of a set of five objects.

Another example that illustrates the interplay between internal and external representations comes from a recent project in which the first author was involved (Vellom & Pape, 2000). In this project, high school students worked with their teachers to learn to represent complex data sets using a software program called Mathematica®. The students represented data they had downloaded from the Internet that was related to various water pollutants in the local watershed. The students were asked to represent these data with the goal of learning about and communicating complex relationships of natural phenomenon. Through layering the graphs of each pollutant's concentration versus varying flow rates of water, one group of students began to understand the complex relationships between the flow rate of the water and the amount of pollutants a river may carry. Through the comparison of various graphs, which served as external representations of the rate of flow and pollutant concentration, the students were able to understand or form an internal representation for the complex relationships within this natural phenomenon.

These descriptions illustrate some of the ways in which students come to use representation(s) to understand the abstract concepts that are central to mathematics learning. We use the term *representation(s)* to refer to both the internal and external manifestations of mathematical concepts. We write representation(s) with the parenthetical "s" to emphasize that, in many places throughout this article, we are speaking of both the act of representing (the verb, to represent) and the external form of the representation (the noun form). This article presents the insights scholars have gained about representation(s) and how this information can be utilized to develop students' understanding of mathematics. We begin by providing a general definition of representation(s), then we discuss differing conceptualizations of representation(s) and the use of representation(s) in mathematics learning, and finally explore some implications for classroom practice.

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THEORY INTO PRACTICE, Volume 40, Number 2, Spring 2001

0040-5841/2001$1.50

Representation(s) in Mathematics

Within the domain of mathematics, representations may be thought of as *internal*—abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience. On the other hand, representations such as numerals, algebraic equations, graphs, tables, diagrams, and charts are *external* manifestations of mathematical concepts that "act as stimuli on the senses" and help us understand these concepts (Janvier, Girardon, & Morand, 1993, p. 81). Finally, representation also refers to the act of externalizing an internal, mental abstraction.

Once the child in the first example begins to form the meaning of the different representations of *five*, she is able to use these representations when she begins to make comparisons such as "more than" and "less than." The visual image of five may be imagined and may serve as a standard to compare other sets of objects. Figure 1 depicts the interplay between internal and external representations, which facilitate the child's ability to make such comparisons. Both the National Council of Teachers of Mathematics' standards documents (NCTM, 2000) and the National Research Council's science standards (NRC, 1996) call for students to be able to use various forms of representations flexibly to investigate and communicate about real-world phenomena. The pathway toward flexible use of multiple representations in teaching and learning mathematics, however, is challenging.

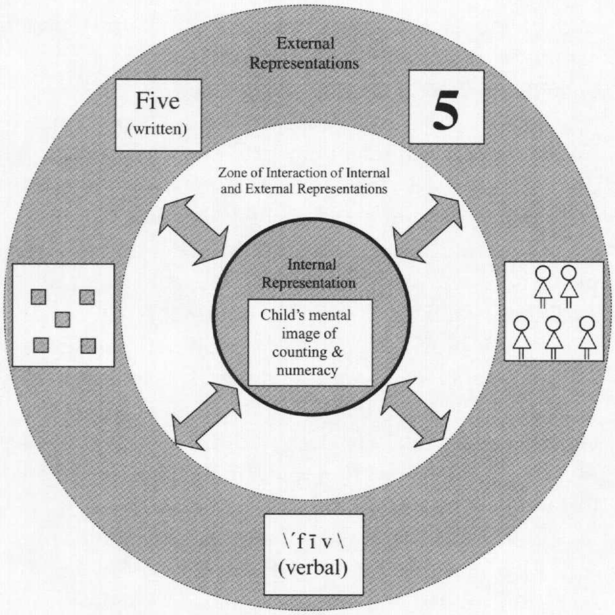

Figure 1. The relationship between internal and external representations in developing child's understanding of the concept of numeracy.

In the recently published *Principles and Standards for School Mathematics* (NCTM, 2000), representation is introduced as a process standard. Although still integral to each of the content standards, this standard has been separated from the individual content standards presented in the initial edition (NCTM, 1989). This shift in prominence is in line with increasing interest in representation(s) among mathematics education researchers (e.g., Maher & Speiser, 1998a, 1998b). Within the NCTM (2000) document, "the term representation refers both to process and to product . . . to the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67). The new process standard calls for all students to be able to:

1. Create and use representations to organize, record, and communicate mathematical ideas;
2. Select, apply, and translate among mathematical representations to solve problems; and
3. Use representation(s) to model and interpret physical, social, and mathematical phenomena. (p. 67)

It is important to note that the representations referred to in each of these statements may be considered as internal, cognitive schemata or the externalizations of these mental constructs. That is, students may formulate internal representations to organize mathematical ideas or to solve problems. Alternatively, they may produce external representations to carry out the same processes.

In the sections that follow, we discuss several issues related to representation(s) in mathematics education. We advocate the position that the development of students' representational thinking is a two-sided process, an interaction of internalization of external representations and externalization of mental images (Figure 1). This interaction often takes place within social interaction. Mathematical concepts are learned through the gradual building of mental images for primary concepts such as the number of objects in a set or complex natural phenomena such as the relationship between flow rate of water and amount of pollutants. The external representations of these concepts—the numeral that represents this number of objects or the graph of the pollutants—stimulate the associated mental image of the number of objects in the set or the relationship within nature.

We argue that there is a mutual influence between the two forms of representations: the nature of an external representation influences the nature of the internal one, and vice versa. Thus, simplistic graphs engender simplistic understandings while complex external representations, such as the layered graph produced by the high school students in the second example above, facilitate understanding of more complex phenomena. Finally, we argue that representation is an inherently social activity. When students are asked to represent data in a graph, the graph should not be a static end result, a picture, but rather a vehicle for discussion to help them establish a relationship or form a justification within a social context. Therefore, we consider *representational thinking* as the learner's ability to interpret, construct, and operate (communicate) effectively with both forms of representations, external and internal, individually and within social situations. We then explore the implications of theory and research for classroom practice involving the social construction of shared meanings of mathematical representations.

A Conceptualization of Representation(s)

Four main ideas characterize many of the various conceptualizations of representations. First, representations may be thought of broadly as mental states. These internal representations are mental images of, for example, a set of five objects. Second, representations may more narrowly be thought of as "mental reproduction of a former mental state" (Seeger, 1998, p. 311). Here, the numeral, 5, or the number, *five*, are examples. Finally the last two formulations include "a structurally equivalent 'presentation' through pictures, symbols or signs," and "something 'in place of' something else" (Seeger, 1998, p. 311). The layered graph that the students used to understand and discuss the relationship between the flow rate of a river and the amount of pollutants in the water is an illustration of these conceptualizations.

Perkins and Unger's (1994) definition of representations includes "symbols in any symbol system (formal notations, language, picturing, etc.) that serve to denote or to exemplify" (p. 2). Symbols and symbol systems support the cognitive activity by reducing the cognitive load (i.e., by reducing all that the individual must think about to accomplish a task), clarifying the problem space, and revealing immediate implications. Thus, symbols or symbol systems help the individual to solve a problem or provide an explanation, prediction, or justification (Perkins & Unger, 1994, pp. 6-7).

It is now well accepted that the use of particular modes of representations (e.g., visual or concrete) leads to improvement of students' mathematical abilities and development of their advanced problem solving and reasoning skills (Krutetskii, 1976; Yakimanskaya, 1991; Presmeg, 1999). That is, the use of multiple representations facilitates students' development of mathematical concepts (e.g., Brenner et al., 1997) and their efforts to carry out tasks such as problem solving (e.g., Greeno & Hall, 1997).

At the same time, in the fields of psychology, pedagogy, and mathematics education, there is an ongoing debate about *how the mind operates* with representations. Accordingly, there is controversy regarding the degree to which the learner is able to extract the mathematical concept from the representation (e.g., concrete material) in which it is embedded. For example, when teachers use base-ten blocks to help children learn the basic procedure for regrouping while adding large numbers, do children readily "see" the connection between the concrete materials and the arithmetic operation?

Figure 2 depicts the addition of 125 and 238 using base-ten blocks. In terms of these materials, each ten-by-ten block or "flat" represents 100, each one-by-ten block or "rod" represents 10, and each "single" represents one. In order for children to operate with these materials, they must align the concrete materials with their mental representation of each number and the operation that is depicted through the manipulation of these materials. Next, they must manipulate the materials by combining the "flats" resulting in three flats or 300, the "rods" resulting in five tens or 50, and the "singles" resulting in 13 singles or 13. Then, they must exchange 10 singles for one "rod" resulting in six tens and three singles or 63. From a pedagogical

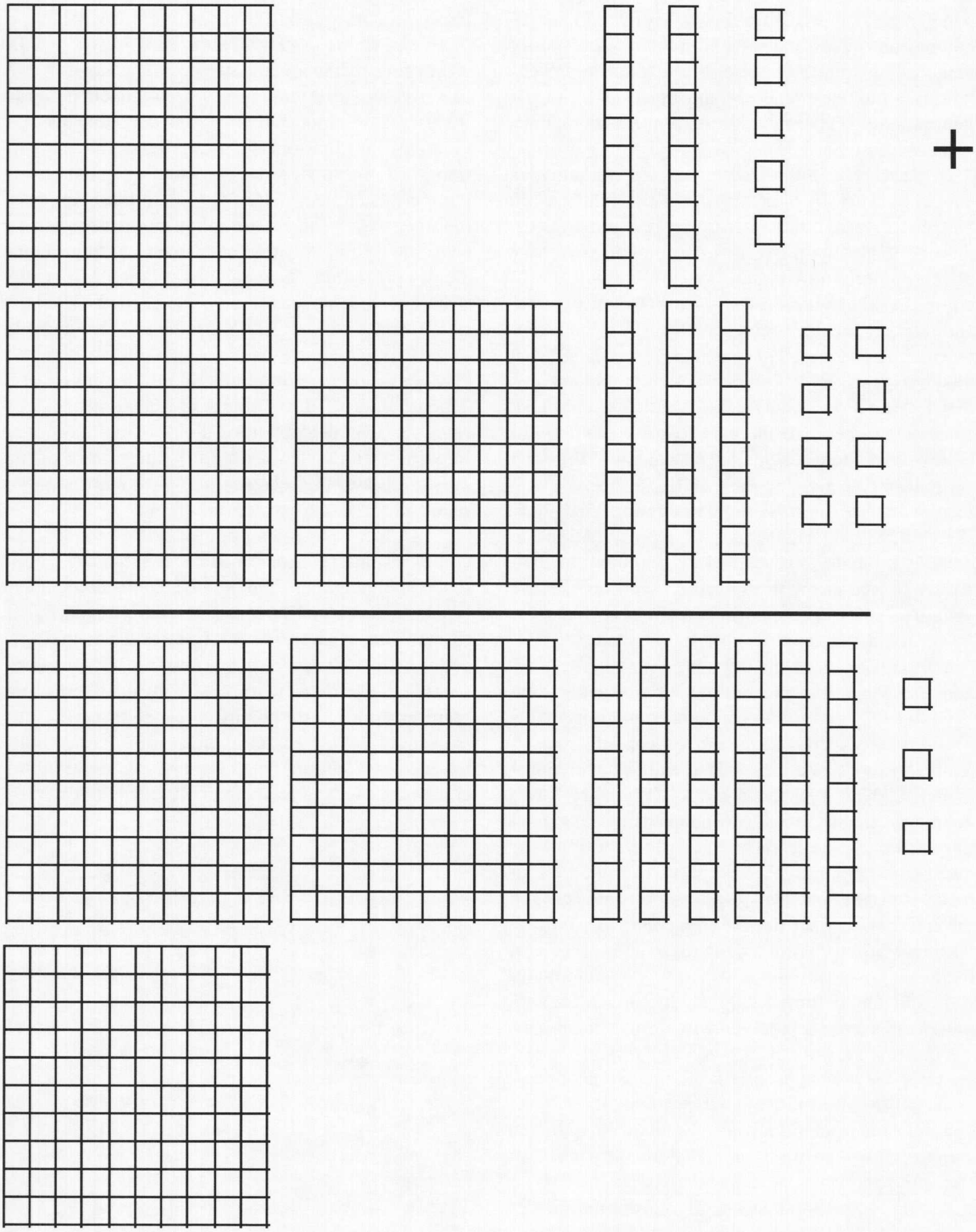

Figure 2. Use of Base-ten blocks for multidigit addition.

perspective, the question of what instructional behaviors are necessary for children to successfully make this mapping between the external representations (concrete materials and numerals) and the internal representation (addition procedure) remains unanswered.

Researchers in mathematics and mathematics education vary in their views on the relationship between external and internal representations. On the one hand, the advocates of a "picture" theory of representation (Mitchell, 1994; Wileman, 1980) argue that there is no difference between external and internal (mental) representations: a mental representation is equivalent to what it represents. From this perspective, the student readily understands that a "flat" represents 100. On the other hand, some researchers (Arnheim, 1969; Mc Kim, 1972) believe that the development of students' thinking is directly connected to their ability to operate with mental images (e.g., seeing, imagining, and idea-sketching). That is, the use of representations to develop students' understanding is related to their ability to operate with the representations (i.e., to visualize with representations themselves).

Based on their critique of the "picture" assumption (i.e., that a mental representation is equivalent to what it represents), Cobb, Yackel, and Wood (1992) claim that this representational view begins with experts' ideas and conceptions and attempts to reproduce these ideas within instructional materials such as base-ten blocks. Therefore, when learning a mathematical procedure using manipulative materials, the learner's task is to construct the necessary mapping between the manipulation of these concrete materials, the base-ten blocks, and the internal abstraction, the procedure for addition with regrouping. From a constructivist perspective, the necessary mapping between the concrete materials and the arithmetic algorithm (procedure) requires intensive social co-construction of meanings. Teachers and students co-construct their understanding of the steps of the mathematical operation while manipulating the materials. These concrete manipulations must be mapped onto the steps of the algorithm such that the learners abstract the sequence of steps.

The conceptualization of representation developed in this article is also based on recent findings in theory of cognition and brain investigation (Caine & Caine, 1994; Chabris & Kosslyn, 1998). According to these studies, the brain works more effectively while making representational patterns for encoding (internalizing) and decoding (externalizing) information. For example, if you ask students to memorize the following multi-digit number, 1123581321345589, it is almost impossible unless they follow the Fibonacci pattern where each succeeding term is the sum of the two immediately preceding. "Seeing" this relationship means that the students can easily internalize ("memorize") and externalize ("reproduce") the number based on the pattern. Another example will help to clarify this point. For a human brain to encode and decode a set of dots indicating the same number (e.g., 20), the placement of the dots in a pattern is important. In case B of Figure 3, the pattern of the dots facilitates the recognition of the number more readily than in case A of this figure.

Unfortunately, as opposed to the varied and complex patterns generated in the human brain, most mathematical content offered to students is typically presented in abstract/symbolic and linear forms. That is, we often attempt to teach the procedure for addition with regrouping without connecting these steps with their physical, concrete manifestations. The cognitive capacity of the human brain, however, more closely resembles multiple representational patterning: combinations of concrete, visual, and abstract. It seems reasonable

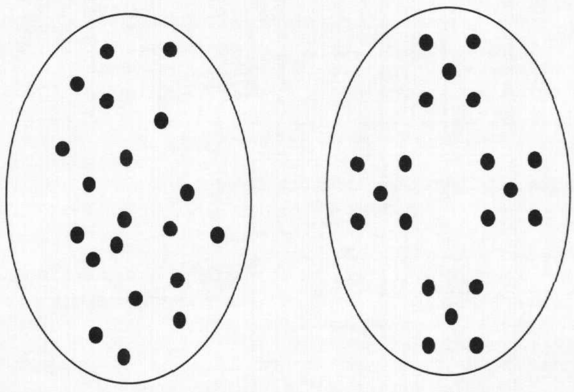
A B

Figure 3. Two different ways of presenting a set of 20 dots.

that the language of the brain consists of multiple representations. Therefore, the development of students' thinking skills requires a multiple representational approach. These ideas are built on and supported by a number of previous investigations of multiple representations in teaching and learning of mathematics (Brenner et al., 1997; Bruner, 1966; Lesh, Post, & Behr, 1987; Wheatley, 1997).

Representation(s) in Learning Mathematics

Although both NCTM (2000) and NRC (1996) documents call for students to be able to use various forms of representations flexibly to investigate and communicate about real-world phenomena, students have great difficulty developing fluency in using representational forms in mathematics. Hiebert (1988) proposes several essential components that facilitate attaining competence in the use of mathematical symbols. First, students must connect individual symbols with the objects they represent. That is, in order to carry out the addition of two numbers or two sets of five and six objects, the symbols "5" and "6" must be aligned with the number of objects in a set and the addition symbol, "+," must be connected to the operation of joining these two sets. Second, they must develop symbol manipulation procedures or the algorithms that govern the use of these symbols. Next, these procedures must become routine. The children must gain flexibility in using the addition algorithm. Finally, these symbols and rules must be elaborated and used as referents for building more abstract systems. For example, addition of whole numbers may be used as a basis for addition of fractions.

This sequence of stages is similar for other representational forms such as graphs. When students are learning to create graphical representations of data, they must learn the conventions of the graphical form and be able to manipulate the symbols for a given type of data set. For example, the high school students discussed in our example needed to understand ideas of scale in order to be able to layer the different pollutants onto the same coordinate axes in a meaningful way. This allowed them to use the graphing conventions simultaneously in a complex situation that enabled them to examine a complex natural phenomenon. Because the students were able to change each of the pollutants to a common scale, they were able to represent their relative concentrations and to make statements that depicted the amounts of pollutants relative to the flow rate of the stream.

Based on the premise that it is unclear how concrete representations assist in this process, Hall (1998) developed the "procedural analogy theory." Accordingly, classroom mathematics scenarios using concrete materials typically begin with exploration followed by more systematic manipulation of the materials. Students are provided some time to explore the materials without direction, but they are frequently directed into a particular use of the materials before their insights are examined. Although there may be some allowance for student-generated ways of using the materials, the teaching sequence typically moves toward standard uses, requiring students' manipulation of the concrete materials to mimic standard algorithms. Thus, students begin joining sets of counters in relation to a mathematical word problem, effectively modeling the action of addition, but are too quickly led to use the materials in a teacher directed fashion. Finally, students are funneled into the written, symbolic procedures.

This sequence follows Bruner's (1966) learning model based upon three levels of engagement with representations: enactive (e.g., manipulating concrete materials), iconic (e.g., pictures and graphs), and symbolic (e.g., numerals). Through early exploration of concrete materials, students are expected to abstract mathematical procedures that are analogous to symbolic procedures. Through the use of analogy, transformation, and simplification, new understandings are built from existing knowledge. That is, the students must map or transform the manipulation of the materials onto the symbolic steps of the mathematical operation and simplify these manipulations through the use of the conventional algorithm to become competent in using this operation. This process is successful only to the degree that the concrete material procedures are analogous to procedures with symbols and the degree to which this connection is made explicit for the learner.

As discussed before, critics of the "representational view of mind" (Cobb et al., 1992) contest this formulation as problematic because instructional materials said to embody mathematical concepts are developed by experts, embody experts' conceptions of mathematical ideas, and may not be readily available or understandable to the novice. Only when the use of representation(s) is built up in the classroom as a cultural activity are students able to come to an understanding of the meanings of the concrete materials and the associated symbolism. That is, in order for the connection to be made between external representations and the mathematical concept they represent and between the mathematics and the children's experiences, representations must be viewed as vehicles for exploration within social contexts that allow for multiple understandings of mathematical content (Seeger, 1998). This conceptualization necessitates an alternative view of the use of representation from that which is typical within mathematics classrooms.

Using Representation(s) in the Classroom

In this section, we highlight the implications for mathematics classroom instruction of the conceptualization of representation developed in this article. This discussion draws upon research related to children's learning of mathematics and the representations they produce for mathematical problems. Finally, this discussion provides suggestions for the classroom teacher.

Representation(s) in social contexts

Students need to practice the use of multiple representations in various situations. Practicing representation(s) must be part of a social environment: "Learning to construct and interpret representations involves learning to participate in the complex practices of communication and reasoning in which the representations are used" (Greeno & Hall, 1997, pp. 361-362). While practicing the construction and use of representational forms, students negotiate the meanings of the forms they have produced as well as the meanings of standard representational forms (DiSessa, Hammer, Sherin, & Kolpakowski, 1991; Greeno & Hall, 1997).

Students' initial attempts to portray phenomena using representations often involve non-standard symbolism that is negotiated and refined through discourse with peers and teachers (DiSessa et al., 1991). In DiSessa and colleagues' work, children represented an object's motion with successive tick marks (or tallies). The object's speed was represented by the angle of the individual mark. This intuitive representation of speed is analogous to the concept of the slope of a distance-time graph for the motion of the object.

Students use their prior knowledge to make sense of all forms of representations. In addition, these initial representations and their evolution depend upon the purpose for creating the representational form, the discussion that surrounds the presentation of these forms, and instructional practices in which the students are engaged. Meira (1995) investigated the transformation of "material displays" (e.g., pictures, written symbols, or tables of data) through activity. Her results indicate that

> (a) the design of displays during problem solving shapes one's mathematical activity and sense making in crucial ways, and (b) knowledge of mathematical representations is not simply recalled and applied to problem solving but also emerges (whether constructed anew or not) out of one's interaction with the social and material settings of activity. (p. 269)

Thus, when the goal of instruction is to learn to represent mathematical concepts or to solve problems involving mathematical representations, students must be given the opportunity to interact with one another and the teacher. Through this interaction within problem-solving situations, knowledge of mathematical representation(s) and mathematical understanding emerges and develops.

Representations as cognitive tools

Representations must be thought of as tools for cognitive activity rather than products or the end result of a task. For example, the models (e.g., graphs or other pictorial representations) produced may be used to help students explain or justify an argument. "When representations are used as tools for understanding and communication, they are constructed and adapted for the purposes at hand" (Greeno & Hall, 1997, p. 362). Representations allow individuals to track intermediary results, ideas, and inferences. Since an external representation embodies the important relationships presented in data or a word problem, they lighten the cognitive load of the individual and serve to organize the individual's further work on a problem. Given the representation, the learner may work on alternative parts of the problem. Representations then may be used to facilitate an argument and to support conclusions.

Thus, in situations that may be characterized as typical school activities (i.e., those in which representations are seen as end results rather than tools for explanation), students often produce representations that lack meaning and from which no relational statements may be drawn. However, in more realistic learning contexts, students may make sense of complex phenomena through their efforts to construct and through the use of graphical representations of these complex systems (Vellom & Pape, 2000). In the study of high school students' representations discussed in the introduction, students were also given a task that, unlike the flow rate task, was similar to a typical school task. They were asked to form representations of data they had collected related to their peers (e.g., eye color, height, and weight). In this activity, students were unable to formulate meaningful relational statements from their data. By contrast, in the real-world activity (analyzing the relationship between flow rate and concentration of pollutants), they came to understand and were able to communicate interesting relationships.

Instructional practice

Finally, students must be taught using a combination of instructional practices. Tchoshanov (1997) conducted a pilot experiment with Russian high school students on trigonometric problem solving and proof. The first comparison group of students ("pure-analytic") was taught by a traditional analytic (algebraic) approach to trigonometric problem solving and proof. The second comparison group ("pure-visual") was taught by a visual (geometric) approach using enactive (i.e., geoboard as manipulative aid) and iconic (pictorial) representations. The third, experimental group ("representational") was taught by a combination of analytic and visual means using translations among different representational modes. The representational group scored 26 percent higher than the visual and 43 percent higher than the analytic groups.

In this experiment, students in the "pure" (analytic and/or visual) groups "stuck" to one particular mode of representation; they were reluctant to use different representations. For instance, students in the pure-visual group tried to avoid any analytic solutions: they were "comfortable" only if they could use visual (geometric) problem-solving and proof techniques. Students in the representational group were much more flexible "switching" from one mode of representation to another in search of better understanding of mathematical concepts. Therefore, we realized that any intensive use of only one particular mode of representation does not improve students' conceptual understanding and representational thinking.

This pilot experiment also showed the importance of students' social interaction using different models (e.g., concrete, visual, and abstract) in the process of developing their representational thinking. We have observed that when multiple representations are used, the level of students' interaction automatically goes up (i.e., they are more eager to exchange their ideas using different representations and they learn from each others' way of solution). This is in contrast to the comparatively low level of interaction when using any single representation.

The main focus in implementing this approach in the actual mathematics classroom was improvement of students' representational thinking in the context of:

- students' exploration of alternative ways of mathematical inquiry and reasoning;
- involvement of students in a variety of hands-on and minds-on activities (e.g., modeling, drawing, imagining, mapping, etc.) in the process of interpreting and communicating mathematical ideas;
- students' construction and co-construction (i.e., within social interaction) of non-standard multiple representations of problem solving and proof techniques; and
- students' understanding of harmonic relationship between different forms of multiple representation of mathematical knowledge.

These examples demonstrate that the development of students' mathematical understanding and representational thinking requires the combination of multiple representations as well as the interaction of both internal and external representations. Through activity, the learner begins to abstract meaning. However, we must be cautious not to advocate the position that this abstraction occurs solely within the individual. It is through the externalization of these abstractions within social environments that learners begin to negotiate the meanings of their understandings and refine these representations accordingly.

The educational significance of this conceptualization is in presenting an alternative holistic approach to the development of representational thinking through construction of students' understanding (internalization) and improvement of their creativity (externalization). Unlike previous studies (e.g., Herscovics, 1996; Hiebert & Carpenter, 1992), which paid attention primarily to the internalization stage, this approach is characterized by its completeness and orientation toward creativity through understanding.

We firmly believe that, in the development of students' representational thinking, internalization without externalization is non-holistic and incomplete. We call the interrelated processes of internalization and externalization *cognitive representation*. The important point here is that despite a tacitly accepted one-sided view of internal representation as a cognitive one (Seeger, 1998), we consider cognitive representation as a zone of interaction of external and internal representations (refer to Figure 1). Consistent with the vision of representation within the NCTM (2000) document, cognitive representation reflects both the process (internalization) and the products (externalization) of representational thinking.

Conclusions

If one of the goals of mathematics is the flexible use of representation(s) (NCTM, 2000), then teachers' behaviors and classroom norms will be important to examine. We would like to highlight four implications of this discussion for classroom practice. First, students must be given the opportunity to practice representation—both the production of external representations and the internalization of mathematical ideas through social activity involving various external representations. Second, representation is inherently a social activity. Students come to understand both the process of representation and its products through social activity.

Third, in order for children to become competent mathematicians, instruction must use a variety of techniques (e.g., analytic and geometric). Finally, representations must be thought of as tools for thinking, explaining, and justifying. Thus, teachers and students must develop classroom norms that facilitate explanation and justification and the use of representations in the service of supporting arguments. Although the pathway is challenging, these insights will facilitate the changes necessary for significant change in classroom practices leading to meaningful mathematical understanding.

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